Tutorial Note XIII

1 An Application of the Riemann-Lebesgue Lemma: the Dirichlet Integral

Ih this section, as an application of the Riemann-Lebesgue lemma, we calculate the Dirichlet integral

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \lim_{R\to\infty}\int_0^R \frac{\sin x}{x}\,\mathrm{d}x.$$

We first note that the Dirichlet kernel:

$$D_N(x) = \sum_{n=-N}^{N} \mathrm{e}^{inx} = \frac{\sin\left(N+1/2\right)x}{\sin(x/2)},$$

satisfies that

$$\int_{-\pi}^{\pi} D_N(x)\,\mathrm{d}x = 2\pi.$$

So

$$\int_{-\pi}^{\pi} \frac{\sin\left(N+1/2\right)x}{\sin(x/2)}\,\mathrm{d}x = 2\pi.$$

If we replace $\sin(x/2)$ by $x/2$, then

$$\int_{-\pi}^{\pi} \frac{\sin\left(N+1/2\right)x}{x/2}\,\mathrm{d}x = 2\int_{-(N+1/2)\pi}^{(N+1/2)\pi} \frac{\sin x}{x}\,\mathrm{d}x.$$

So if we let $N\to\infty$,

$$\lim_{N\to\infty}\int_{-\pi}^{\pi} \frac{\sin\left(N+1/2\right)x}{x/2}\,\mathrm{d}x = 4\int_0^\infty \frac{\sin x}{x}.$$

So it remains to calculate the difference

$$\int_{-\pi}^{\pi} \sin\left(N+1/2\right)x\left(\frac{1}{\sin(x/2)} - \frac{1}{x/2}\right)\mathrm{d}x.$$

Note that

$$\frac{1}{\sin(x/2)} - \frac{1}{x/2} = \frac{x/2 - \sin(x/2)}{\sin(x/2)(x/2)}$$

is continuous on $[-\pi,\pi]$. So we can apply the Riemann-Lebesgue lemma and we have

$$\lim_{N\to\infty}\int_{-\pi}^{\pi} \sin\left(N+1/2\right)x\left(\frac{1}{\sin(x/2)} - \frac{1}{x/2}\right)\mathrm{d}x = 0.$$

Therefore,

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

2 Decay of Fourier Coefficients: Continuation

First, we recall the conclusions about decay of Fourier coefficients listed in the last tutorial:

- If f is α-Hölder continuous, $\hat{f}(n) = O(1/|n|^\alpha)$;
- If f is bounded monotone, $\hat{f}(n) = O(1/|n|)$;
- If f is continuous, $\hat{f}(n) = o(1)$;
- If f is C^k, $\hat{f}(n) = o(1/|n|^k)$.

In this section, we present their proofs. First, we note that the fourth conclusion is just a corollary of the Riemann-Lebesgue lemma since

$$\widehat{f^{(k)}}(n) = (\mathrm{i}n)^k \hat{f}(n).$$

So we mainly prove the first two conclusions here. WLOG, we assume that $n > 0$. To see the cancellations of e^{-inx}, we return to Riemann sums. We have

$$\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x \approx \sum_{k=0}^{N-1} \mathrm{e}^{-\mathrm{i}n\cdot 2k\pi/N} f\left(\frac{2k\pi}{N}\right)\frac{2\pi}{N}. \tag{1}$$

For the first conclusion, if we take $N = 2nl$, then

$$\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x \approx \sum_{k=0}^{2nl-1} \mathrm{e}^{-\mathrm{i}k\pi/l} f\left(\frac{k\pi}{nl}\right)\frac{\pi}{nl}.$$

An important observation is that

$$\mathrm{e}^{-\mathrm{i}k\pi/l} f\left(\frac{(k+l)\pi}{nl}\right) + \mathrm{e}^{-\mathrm{i}(k+l)\pi/l} f\left(\frac{k\pi}{nl}\right) = \mathrm{e}^{-\mathrm{i}k\pi/l}\left[f\left(\frac{(k+l)\pi}{nl}\right) - f\left(\frac{k\pi}{nl}\right)\right].$$

Inspired by this, since

$$\sum_{k=0}^{2nl-1} \mathrm{e}^{-\mathrm{i}k\pi/l} f\left(\frac{k\pi}{nl}\right)\frac{\pi}{nl} = \sum_{k=0}^{2nl-1} \mathrm{e}^{-\mathrm{i}(k+l)\pi/l} f\left(\frac{(k+l)\pi}{nl}\right)\frac{\pi}{nl},$$

we have

$$\begin{aligned}
\sum_{k=0}^{2nl-1} \mathrm{e}^{-\mathrm{i}k\pi/l} f\left(\frac{k\pi}{nl}\right)\frac{\pi}{nl} &= \frac{1}{2}\sum_{k=0}^{2nl-1}\left[\mathrm{e}^{-\mathrm{i}k\pi/l} f\left(\frac{(k+l)\pi}{nl}\right) + \mathrm{e}^{-\mathrm{i}(k+l)\pi/l} f\left(\frac{k\pi}{nl}\right)\right]\frac{\pi}{nl} \\
&= \frac{1}{2}\sum_{k=0}^{2nl-1} \mathrm{e}^{-\mathrm{i}k\pi/l}\left[f\left(\frac{(k+l)\pi}{nl}\right) - f\left(\frac{k\pi}{nl}\right)\right]\frac{\pi}{nl}.
\end{aligned}$$

So

$$\left|\sum_{k=0}^{2nl-1} \mathrm{e}^{-\mathrm{i}k\pi/l} f\left(\frac{k\pi}{nl}\right)\frac{\pi}{nl}\right| \le \pi K\left(\frac{\pi}{n}\right)^{\alpha},$$

provided that $|f(x)-f(y)| \le K|x-y|^{\alpha}$. According to this idea, a neat proof is as follows:

$$\begin{aligned}\left|\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x\right| &= \frac{1}{2}\left|\int_0^{2\pi}[f(x)-f(x+\pi/n)]\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x\right| \\ &\le \pi K\left(\frac{\pi}{n}\right)^{\alpha}.\end{aligned}$$

Moreover, this result is sharp for $\alpha \in (0,1)$. Consider the lacunary Fourier series:

$$\sum_{k=0}^{\infty} 2^{-k\alpha}\mathrm{e}^{\mathrm{i}2^k x}$$

for $\alpha \in (0,1)$. It is obvious that its Fourier coefficients are $O(1/|n|^{\alpha})$. Next we show that it is α-Hölder continuous. Since

$$|\mathrm{e}^{\mathrm{i}2^k x}-\mathrm{e}^{\mathrm{i}2^k y}| \le 2 \wedge 2^k|x-y|,$$

$$\begin{aligned}\left|\sum_{k=0}^{\infty} 2^{-k\alpha}\left(\mathrm{e}^{\mathrm{i}2^k x}-\mathrm{e}^{\mathrm{i}2^k y}\right)\right| &\le \sum_{k=0}^{\infty} 2^{-k\alpha}\left(2\wedge 2^k|x-y|\right) \\ &\lesssim |x-y|^{\alpha}.\end{aligned}$$

Next we prove the second conclusion. For (1), by summation by parts, we have

$$\begin{aligned}\sum_{k=0}^{N-1} \mathrm{e}^{-\mathrm{i}n\cdot 2k\pi/N} f\left(\frac{2k\pi}{N}\right)\frac{2\pi}{N} = \sum_{k=0}^{N-2} S_k\left[f\left(\frac{2k\pi}{N}\right)-f\left(\frac{2(k+1)\pi}{N}\right)\right]\frac{2\pi}{N} \\ + S_{N-1} f\left(\frac{2(N-1)\pi}{N}\right)\frac{2\pi}{N},\end{aligned}$$

where

$$S_k = \sum_{l=0}^{k} \mathrm{e}^{-\mathrm{i}n\cdot 2l\pi/N}.$$

Note that

$$S_k = \frac{\mathrm{e}^{-\mathrm{i}n\cdot 2(k+1)\pi/N}-1}{\mathrm{e}^{-\mathrm{i}n2\pi/N}-1}.$$

We have

$$|S_k| \le \frac{1}{\sin(n\pi/N)}.$$

So

$$\left|\sum_{k=0}^{N-1} \mathrm{e}^{-\mathrm{i}n\cdot 2k\pi/N} f\left(\frac{2k\pi}{N}\right)\frac{2\pi}{N}\right| \le \frac{2\pi}{\sin(n\pi/N)N}\cdot 3M,$$

provided that $|f| \le M$. Let $N \to \infty$, then we obtain that

$$\left|\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x\right| \le \frac{6M}{n}.$$

3 Bernstein Theorem

In this section, we prove the Bernstein theorem.

Theorem 3.1 (Bernstein)

If f is α-Hölder continuous, where $\alpha > 1/2$, then $\sum_n |\hat{f}(n)| < \infty$.

Proof. By the above conclusions about decay of Fourier coefficients, we have $\hat{f}(n) = O(1/|n|^\alpha)$, which is not enough to prove $\sum_n |\hat{f}(n)| < \infty$. So we need more delicate estimates. Our tool is Parseval's identity. To exploit the regularity of f, we consider $f(x+h) - f(x)$.

$$[f(x+h) - f(x)]\hat{}(n) = (\mathrm{e}^{inh} - 1)\hat{f}(n).$$

By Parseval's identity,

$$\frac{1}{2\pi}\int_0^{2\pi} [f(x+h) - f(x)]^2 \,\mathrm{d}x = \sum_n |\mathrm{e}^{inh} - 1|^2 |\hat{f}(n)|^2.$$

If $h = \pi/2^k$,

$$|\mathrm{e}^{inh} - 1|^2 = 4\sin^2\left(\frac{n\pi}{2^{k+1}}\right) \geq 2$$

for n satisfying $2^{k-1} < |n| \leq 2^k$. So

$$2 \sum_{2^{k-1}<|n|\leq 2^k} |\hat{f}(n)|^2 \leq K^2 \left(\frac{\pi}{2^k}\right)^{2\alpha},$$

provided that $|f(x) - f(y)| \leq K|x-y|^\alpha$. By the Cauchy-Schwarz inequality,

$$\sum_{2^{k-1}<|n|\leq 2^k} |\hat{f}(n)| \leq 2^{k/2} \cdot \frac{K}{\sqrt{2}} \cdot \left(\frac{\pi}{2^k}\right)^{\alpha} \leq K\pi^\alpha 2^{(1/2-\alpha)k}.$$

Since $\alpha > 1/2$, we have

$$\sum_n |\hat{f}(n)| < \infty.$$

□

The contents of this tutorial note are mainly from the exercises of chapter 3 of Stein's Fourier analysis.